

# Strong solutions for a stochastic model of 2-D second grade fluids driven by Lévy noise

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**Abstract:** We consider a stochastic model of incompressible non-Newtonian fluids of second grade on a bounded domain of  $\mathbb{R}^2$  driven by Lévy noise. Applying the variational approach, global existence and uniqueness of strong probabilistic solution is established.

**Key Words:** Second grade fluids; Non-Newtonian fluids; Stochastic partial differential equations; Locally monotone condition; Lévy noise.

## 1 Introduction

In petroleum industry, polymer technology and problems of liquid crystals suspensions, non-Newtonian fluids of differential type often arise (see [30]). It has attracted much attention from a theoretical point of view, since the classical theory of Newtonian fluids is unable to explain properties observed in the nature. In this article, we are interested in a special class of non-Newtonian fluids of differential type, namely the second grade fluids, which is an admissible model of slow flow fluids such as industrial fluids, slurries, polymer melts, etc.

Mathematically, the stochastic models for incompressible second grade fluids are described by the following equation:

$$\left\{ \begin{array}{l} d(u(t) - \alpha \Delta u(t)) + \left( -\mu \Delta u(t) + \operatorname{curl}(u(t) - \alpha \Delta u(t)) \times u(t) + \nabla \mathfrak{P} \right) dt \\ = F(u(t), t) dt + \sigma(u(t), t) dW(t) + \int_{\mathbb{Z}} f(u(t-), t, z) \tilde{N}(dz dt), \quad \text{in } \mathcal{O} \times (0, T], \\ \operatorname{div} u = 0 \quad \text{in } \mathcal{O} \times (0, T]; \\ u = 0 \quad \text{in } \partial \mathcal{O} \times [0, T]; \\ u(0) = \xi \quad \text{in } \mathcal{O}, \end{array} \right. \quad (1.1)$$

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where  $\mathcal{O}$  is a bounded domain of  $\mathbb{R}^2$  with boundary  $\partial\mathcal{O}$  of class  $\mathcal{C}^{3,1}$ ;  $u = (u_1, u_2)$ ,  $\mathfrak{P}$  represent the random velocity and modified pressure respectively;  $F(u(t), t)$  is the external force;  $W$  is a one-dimensional standard Brownian motion;  $\tilde{N}$  is a compensated Poisson random measure.

The second grade fluids has properties of boundedness, stability and exponential decay (see [10]), and has interesting connections with many other fluid models, see [34, 10, 9, 11, 3, 31] and references therein. For example, as shown in [19], the second grade fluids reduces to the Navier-Stokes equation(NSE) when  $\alpha = 0$ , and it's also a good approximation of the NSE. We refer the reader to [5], [4], [24] for a comprehensive theory of the deterministic second grade fluids.

On the other hand, in recent years, introducing a jump-type noises such as Lévy-type or Poisson-type perturbations has become extremely popular for modeling natural phenomena, because these noises are very nice choice to reproduce the performance of some natural phenomena in real world models, such as some large moves and unpredictable events. There is a large amount of literature on the existence and uniqueness solutions for stochastic partial differential equations(SPDEs) driven by jump-type noises. We refer the reader to [7, 1, 8, 6, 35, 37, 38, 2, 16].

We particular mention that for the stochastic 2-D second grade fluids driven by pure jump noises, Hausenblas, Razafimandimby and Sango [17] obtained the global existence of a martingale solution, that is the weak solution in the probabilistic sense. The purpose of this paper is to establish the global existence and uniqueness of strong probabilistic solutions for stochastic 2-D second grade fluids driven by general Lévy noises. To obtain our result, we will use the so called variational approach. This approach was initiated by Pardoux [25] and Krylov and Rozovskii [20], then further developed by many authors e.g. see [14, 13, 29, 26, 21, 2, 22]. It has been proved to be a powerful tool to establish the well-posedness for stochastic dynamical systems with locally monotone and coercive coefficients. However, the existing results in the literature could not cover the situation considered in this paper, because the second grade fluids do not satisfy the coercivity condition required. Under our consideration, we prove the main result by three steps: we first establish some non-trivial a priori estimates of the Galerkin approximations, then we show the limit of those approximate solutions solves the original equation by applying the monotonicity arguments, finally we prove the uniqueness of solutions.

Finally, we mention several other results concerning the study of the stochastic models for two-dimensional second grade fluids driven by Wiener processes. In [27, 28]. the authors established the global existence and uniqueness of strong probabilistic solutions, and they also investigated the long time behavior of the solution and the asymptotic behavior of the solutions. Large deviations and moderate deviations for the stochastic models of second grade fluids driven by Wiener processes have been established by in [39] and [40] respectively. The exponential mixing property of the stochastic models of second grade fluids was established in [36].

The organization of this paper is as follows. In Section 2, we introduce some preliminaries and state some lemmas which will be used later. In Section 3, we formulate the hypotheses and state our main result. Section 4 is devoted to the proof of the main result.

Throughout this paper,  $C, C_1, C_2 \dots$  are positive constants whose value may be different from line to line.

## 2 Preliminaries

In this section, we will describe the framework in details, and introduce some functional spaces and preliminaries that are needed in the paper.

Let  $(\Omega, \mathcal{F}, \{\mathcal{F}_t\}_{t \in [0, T]}, P)$  be a given complete filtered probability space, satisfying the usual conditions. Let  $(Z, \mathcal{Z})$  be a measurable space, and  $\nu$  a  $\sigma$ -finite measure on it. For  $B \in \mathcal{Z}$  with  $\nu(B) < \infty$ , we write

$$\tilde{N}((0, t] \times B) := N((0, t] \times B) - t\nu(B), \quad t \geq 0,$$

for the compensated Poisson random measure on  $[0, T] \times \Omega \times Z$ , where  $N$  is a Poisson random measure on  $[0, T] \times Z$  with intensity measure  $dt \times \nu(dz)$ . Assume that  $W$  is a one-dimensional standard Brownian motion on the probability space  $(\Omega, \mathcal{F}, \{\mathcal{F}_t\}_{t \in [0, T]}, P)$ .

To formulate the equation, we introduce the following spaces. For  $p \geq 1$  and  $k \in \mathbb{N}$ , we denote by  $L^p(\mathcal{O})$  and  $W^{k,p}(\mathcal{O})$  the usual  $L^p$  and Sobolev spaces over  $\mathcal{O}$  respectively, and write  $H^k(\mathcal{O}) := W^{k,2}(\mathcal{O})$ . Let  $W_0^{k,p}(\mathcal{O})$  be the closure in  $W^{k,p}(\mathcal{O})$  of  $\mathcal{C}_c^\infty(\mathcal{O})$  the space of infinitely differentiable functions with compact supports in  $\mathcal{O}$ , and denote  $W_0^{k,2}(\mathcal{O})$  by  $H_0^k(\mathcal{O})$ . We equip  $H_0^1(\mathcal{O})$  with the scalar product

$$((u, v)) = \int_{\mathcal{O}} \nabla u \cdot \nabla v dx = \sum_{i=1}^2 \int_{\mathcal{O}} \frac{\partial u}{\partial x_i} \frac{\partial v}{\partial x_i} dx,$$

where  $\nabla$  is the gradient operator. It is well known that the norm  $\|\cdot\|$  generated by this scalar product is equivalent to the usual norm of  $W^{1,2}(\mathcal{O})$ .

For any vector space  $X$ , write  $\mathbb{X} = X \times X$ . Set

$$\begin{aligned} \mathcal{C} &= \left\{ u \in [\mathcal{C}_c^\infty(\mathcal{O})]^2 : \operatorname{div} u = 0 \right\}, \\ \mathbb{V} &= \text{the closure of } \mathcal{C} \text{ in } \mathbb{H}^1(\mathcal{O}), \\ \mathbb{H} &= \text{the closure of } \mathcal{C} \text{ in } \mathbb{L}^2(\mathcal{O}). \end{aligned}$$

We denote by  $(\cdot, \cdot)$  and  $|\cdot|$  the inner product in  $\mathbb{L}^2(\mathcal{O})$  (in  $\mathbb{H}$ ) and the induced norm, respectively. The inner product and the norm of  $\mathbb{H}_0^1(\mathcal{O})$  are denoted respectively by  $((\cdot, \cdot))$  and  $\|\cdot\|$ . We endow the space  $\mathbb{V}$  with the norm generated by the following inner product

$$(u, v)_{\mathbb{V}} := (u, v) + \alpha((u, v)), \quad \text{for any } u, v \in \mathbb{V},$$

and the norm in  $\mathbb{V}$  is denoted by  $|\cdot|_{\mathbb{V}}$ . The Poincaré's inequality implies that there exists a constant  $\mathcal{P} > 0$  such that the following inequalities holds

$$(\mathcal{P}^2 + \alpha)^{-1} |v|_{\mathbb{V}}^2 \leq \|v\|^2 \leq \alpha^{-1} |v|_{\mathbb{V}}^2, \quad \text{for any } v \in \mathbb{V}. \quad (2.1)$$

We also introduce the following space

$$\mathbb{W} = \{u \in \mathbb{V} : \operatorname{curl}(u - \alpha \Delta u) \in L^2(\mathcal{O})\},$$

and endow it with the semi-norm generated by the scalar product

$$(u, v)_{\mathbb{W}} := (\operatorname{curl}(u - \alpha \Delta u), \operatorname{curl}(v - \alpha \Delta v)). \quad (2.2)$$

The semi-norm in  $\mathbb{W}$  is denoted by  $|\cdot|_{\mathbb{W}}$ . The following result states that this semi-norm  $|\cdot|_{\mathbb{W}}$  is equivalent to the usual norm in  $\mathbb{H}^3(\mathcal{O})$ . The proof can be found in [5, 4].

**Lemma 2.1** *Set*

$$\widetilde{\mathbb{W}} = \{v \in \mathbb{H}^3(\mathcal{O}) : \operatorname{div} v = 0 \text{ and } v|_{\partial\mathcal{O}} = 0\},$$

*then the following (algebraic and topological) identity holds:*

$$\mathbb{W} = \widetilde{\mathbb{W}}.$$

*Moreover, there exists a constant  $C > 0$  such that*

$$|v|_{\mathbb{H}^3(\mathcal{O})} \leq C|v|_{\mathbb{W}}, \quad \forall v \in \widetilde{\mathbb{W}}. \quad (2.3)$$

If we identify the Hilbert space  $\mathbb{V}$  with its dual space  $\mathbb{V}^*$  by the Riesz representation, then we obtain a Gelfand triple

$$\mathbb{W} \subset \mathbb{V} \subset \mathbb{W}^*.$$

We denote by  $\langle f, v \rangle$  the dual relation between  $f \in \mathbb{W}^*$  and  $v \in \mathbb{W}$  from now on. It is easy to see

$$(v, w)_{\mathbb{V}} = \langle v, w \rangle, \quad \forall v \in \mathbb{V}, \quad \forall w \in \mathbb{W}. \quad (2.4)$$

Note that the injection of  $\mathbb{W}$  into  $\mathbb{V}$  is compact, thus there exists a sequence  $\{e_i\}$  of elements of  $\mathbb{W}$  which forms an orthonormal basis in  $\mathbb{W}$ , and an orthogonal system in  $\mathbb{V}$ , moreover this sequence verifies:

$$(v, e_i)_{\mathbb{W}} = \lambda_i(v, e_i)_{\mathbb{V}}, \quad \text{for any } v \in \mathbb{W}, \quad (2.5)$$

where  $0 < \lambda_i \uparrow \infty$ . Lemma 4.1 in [4] implies that

$$e_i \in \mathbb{H}^4(\mathcal{O}), \quad \forall i \in \mathbb{N}. \quad (2.6)$$

Consider the following “generalized Stokes equations”:

$$\begin{aligned} v - \alpha \Delta v &= f && \text{in } \mathcal{O}, \\ \operatorname{div} v &= 0 && \text{in } \mathcal{O}, \\ v &= 0 && \text{on } \partial\mathcal{O}. \end{aligned} \quad (2.7)$$

The following result can be derived from [32] and also can be found in [27, 28].

**Lemma 2.2** *If  $f \in \mathbb{V}$ , then the system (2.7) has a unique solution  $v \in \mathbb{W}$ , and the following relations hold*

$$(v, g)_{\mathbb{V}} = (f, g), \quad \forall g \in \mathbb{V}, \quad (2.8)$$

$$|v|_{\mathbb{W}} \leq C|f|_{\mathbb{V}}. \quad (2.9)$$

Define the Stokes operator by

$$Au := -\Pi \Delta u, \quad \forall u \in D(A) = \mathbb{H}^2(\mathcal{O}) \cap \mathbb{V}, \quad (2.10)$$

here the mapping  $\Pi : \mathbb{L}^2(\mathcal{O}) \longrightarrow \mathbb{H}$  is the usual Helmholtz-Leray projection. The Helmholtz-Leray projection implies that for any  $u \in \mathbb{L}^2(\mathcal{O})$ , there exists a  $\phi_u \in \mathbb{H}^1(\mathcal{O})$  such that

$$u = \Pi u + \nabla \phi_u,$$

Hence,

$$\operatorname{curl}(\Pi v) = \operatorname{curl}(v) \quad \forall v \in \mathbb{H}^1. \quad (2.11)$$

Therefore,

$$|u|_{\mathbb{W}} := |\operatorname{curl}(I + \alpha A)u|. \quad (2.12)$$

It follows from Lemma 2.2 that the operator  $(I + \alpha A)^{-1}$  defines an isomorphism from  $\mathbb{V}$  into  $\mathbb{W}$ . Moreover, for any  $f, g \in \mathbb{V}$ , the following properties hold

$$\begin{aligned} ((I + \alpha A)^{-1}f, g)_{\mathbb{V}} &= (f, g), \\ |(I + \alpha A)^{-1}f|_{\mathbb{V}} &\leq C(I + \alpha A)^{-1}f|_{\mathbb{W}} \leq C_A|f|_{\mathbb{V}}. \end{aligned} \quad (2.13)$$

Let  $\widehat{A} := (I + \alpha A)^{-1}A$ , then  $\widehat{A}$  is a continuous linear operator from  $\mathbb{W}$  onto itself, and satisfies

$$(\widehat{A}u, v)_{\mathbb{V}} = (Au, v) = ((u, v)), \quad \forall u \in \mathbb{W}, v \in \mathbb{V}, \quad (2.14)$$

hence

$$(\widehat{A}u, u)_{\mathbb{V}} = \|u\|, \quad \forall u \in \mathbb{W}. \quad (2.15)$$

We recall the following estimates which can be found in [28].

**Lemma 2.3** *For any  $u, v, w \in \mathbb{W}$ , we have*

$$|(\operatorname{curl}(u - \alpha \Delta u) \times v, w)| \leq C|u|_{\mathbb{W}}|v|_{\mathbb{V}}|w|_{\mathbb{W}}, \quad (2.16)$$

and

$$|(\operatorname{curl}(u - \alpha \Delta u) \times u, w)| \leq C|u|_{\mathbb{V}}^2|w|_{\mathbb{W}}. \quad (2.17)$$

Defining the bilinear operator  $\widehat{B}(\cdot, \cdot) : \mathbb{W} \times \mathbb{V} \longrightarrow \mathbb{W}^*$  by

$$\widehat{B}(u, v) := (I + \alpha A)^{-1}\Pi(\operatorname{curl}(u - \alpha \Delta u) \times v).$$

We have the following consequence from the above lemma.

**Lemma 2.4** *For any  $u \in \mathbb{W}$  and  $v \in \mathbb{V}$ , it holds that*

$$|\widehat{B}(u, v)|_{\mathbb{W}^*} \leq C|u|_{\mathbb{W}}|v|_{\mathbb{V}}, \quad (2.18)$$

and

$$|\widehat{B}(u, u)|_{\mathbb{W}^*} \leq C_B|u|_{\mathbb{V}}^2. \quad (2.19)$$

In addition

$$\langle \widehat{B}(u, v), v \rangle = 0, \quad \forall u, v \in \mathbb{W}, \quad (2.20)$$

which implies

$$\langle \widehat{B}(u, v), w \rangle = -\langle \widehat{B}(u, w), v \rangle, \quad \forall u, v, w \in \mathbb{W}. \quad (2.21)$$

### 3 Hypotheses and the result

In this section, we will formulate precise assumptions on coefficients and state our main results.

Let

$$\begin{aligned} F &: \mathbb{V} \times [0, T] \longrightarrow \mathbb{V}; \\ \sigma &: \mathbb{V} \times [0, T] \longrightarrow \mathbb{V}; \\ f &: \mathbb{V} \times [0, T] \times Z \longrightarrow \mathbb{V}, \end{aligned}$$

be given measurable maps. We introduce the following notations:

$$\begin{aligned} \widehat{F}(u, t) &:= (I + \alpha A)^{-1} F(u, t); \\ \widehat{\sigma}(u, t) &:= (I + \alpha A)^{-1} \sigma(u, t); \\ \widehat{f}(u, t, z) &:= (I + \alpha A)^{-1} f(u, t, z). \end{aligned}$$

Applying  $(I + \alpha A)^{-1}$  to the equation (1.1), we see that (1.1) is equivalent to the stochastic evolution equation

$$\begin{cases} du(t) + \mu \widehat{A}u(t)dt + \widehat{B}(u(t), u(t))dt \\ \quad = \widehat{F}(u(t), t) + \widehat{\sigma}(u(t), t) dW(t) + \int_Z \widehat{f}(u(t-), t, z) \widetilde{N}(dzdt), \\ u(0) = \xi \quad \text{in } \mathbb{W}. \end{cases} \quad (3.1)$$

Let us formulate assumptions on coefficients. Suppose that there exists constants  $C \geq 0$ ,  $\widetilde{\rho} \in L^1([0, T])$  and  $K \in L^2([0, T])$  such that the following hold for all  $u_1, u_2 \in \mathbb{V}$  and  $t \in [0, T]$ :

**(S1)** (Lipschitz continuity)

$$\begin{aligned} &|F(u_1, t) - F(u_2, t)|_{\mathbb{V}}^2 + |\sigma(u_1, t) - \sigma(u_2, t)|_{\mathbb{V}}^2 + \int_Z |f(u_1, t, z) - f(u_2, t, z)|_{\mathbb{V}}^2 \nu(dz) \\ &\leq \widetilde{\rho}(t) |u_1 - u_2|_{\mathbb{V}}^2. \end{aligned}$$

**(S2)** (Growth condition)

$$|F(u, t)|_{\mathbb{V}}^2 + |\sigma(u, t)|_{\mathbb{V}}^2 + \int_Z |f(u, t, z)|_{\mathbb{V}}^2 \nu(dz) \leq K(t) + C|u|_{\mathbb{V}}^2,$$

and

$$\int_Z |f(u, t, z)|_{\mathbb{V}}^4 \nu(dz) \leq K^2(t) + C|u|_{\mathbb{V}}^4.$$

**Definition 3.1** A  $\mathbb{V}$ -valued càdlàg and  $\mathbb{W}$ -valued weakly càdlàg  $\{\mathcal{F}_t\}$ -adapted stochastic process  $u$  is called a solution of the system (1.1), if the following two conditions hold:

- (1)  $u \in L^2(\Omega \times [0, T]; \mathbb{W})$ ;
- (2) for any  $t \in [0, T]$ , the following equation holds in  $\mathbb{W}^*$   $P$ -a.s.:

$$\begin{aligned} &u(t) + \mu \int_0^t \widehat{A}u(s) ds + \int_0^t \widehat{B}(u(s), u(s)) ds \\ &= \xi + \int_0^t \widehat{F}(u(s), s) ds + \int_0^t \widehat{\sigma}(u(s), s) dW(s) + \int_0^t \int_Z \widehat{f}(u(s-), s, z) \widetilde{N}(dzds). \end{aligned}$$

Now we can state the main result of this paper.

**Theorem 3.2** *Suppose that (S1) and (S2) are satisfied. Then for any  $\xi \in L^4(\Omega, \mathcal{F}_0, P; \mathbb{W})$ , equation (3.1) has a unique solution  $u$ . Moreover,  $u \in L^4(\Omega, \mathcal{F}, P; L^\infty([0, T]; \mathbb{W}))$ .*

## 4 The proof of Theorem 3.2

We divide the proof of the main result into three parts. In Section 4.1, we establish some crucial a priori estimates for the Galerkin approximations. In Section 4.2, we prove the existence. The uniqueness is proved in Section 4.3.

### 4.1 Galerkin approximations

The proof of the existence of solutions of system (3.1) is based on Galerkin approximations. From (2.5) we know that  $\{\sqrt{\lambda_i}e_i\}$  is an orthonormal basis of  $\mathbb{V}$ . Let  $\mathbb{V}_n$  denote the  $n$ -dimensional subspace spanned by  $\{e_1, e_2, \dots, e_n\}$  in  $\mathbb{W}$ . Let  $\Pi_n : \mathbb{W}^* \rightarrow \mathbb{V}_n$  be defined by

$$\Pi_n g := \sum_{i=1}^n \lambda_i \langle g, e_i \rangle e_i, \quad \forall g \in \mathbb{W}^*.$$

Clearly,  $\Pi_n|_{\mathbb{V}}$  is just the orthogonal projection from  $\mathbb{V}$  onto  $\mathbb{V}_n$ .

Now, for any integer  $n \geq 1$ , we seek a solution  $u_n$  to the equation

$$\begin{cases} du^n(t) = \Pi_n \left[ -\mu \hat{A}u^n(t) - \hat{B}(u^n(t), u^n(t)) + \hat{F}(u^n(t), t) \right] dt + \Pi_n \hat{\sigma}(u^n(t), t) dW(t) \\ \quad + \int_Z \Pi_n \hat{f}(u^n(t-), t, z) \tilde{N}(dz dt), \quad t > 0, \\ u_n(0) = \Pi_n \xi, \end{cases} \quad (4.1)$$

such that

$$u^n(t) = \sum_{j=1}^n u_{jn}(t) e_j, \quad t \geq 0,$$

for appropriate choice of real-valued random processes  $u_{jn}$ .

Let

$$\hat{\mathcal{A}}(u, t) := -\mu \hat{A}u - \hat{B}(u, u) + \hat{F}(u, t); \quad (4.2)$$

$$\rho(u, t) := 1 + 2C_B |u|_{\mathbb{W}} + C_A^2 \tilde{\rho}(t). \quad (4.3)$$

**Lemma 4.1** *Under assumptions of Theorem 3.2, for all  $u, u_1, u_2 \in \mathbb{W}$  and any  $t \in [0, T]$ , the following properties (H1), (H2) and (H3) hold:*

(H1) *The map  $s \mapsto \langle \hat{\mathcal{A}}(u_1 + su_2, t), u \rangle$  is continuous on  $\mathbb{R}$ .*

(H2)

$$\begin{aligned} & 2\langle \hat{\mathcal{A}}(u_1, t) - \hat{\mathcal{A}}(u_2, t), u_1 - u_2 \rangle + |\hat{\sigma}(u_1, t) - \hat{\sigma}(u_2, t)|_{\mathbb{V}}^2 + \int_Z |\hat{f}(u_1, t, z) - \hat{f}(u_2, t, z)|_{\mathbb{V}}^2 \nu(dz) \\ & \leq \rho(u_2, t) |u_1 - u_2|_{\mathbb{V}}^2. \end{aligned}$$

(H3)

$$|\widehat{\mathcal{A}}(u, t)|_{\mathbb{W}^*}^2 \leq C(K(t) + |u|_{\mathbb{V}}^2 + |u|_{\mathbb{V}}^4).$$

If  $u \in \mathbb{V}_m$  for some  $m \in \mathbb{N}$ , then  $\widehat{\mathcal{A}}$  can satisfy (H4) for any  $\theta > 0$ :  
(H4) (Coercivity)

$$\langle \widehat{\mathcal{A}}(u, t), u \rangle + \theta |u|_{\mathbb{W}}^2 \leq K(t) + C_{m, \theta} |u|_{\mathbb{V}}^2.$$

Remark: since the  $\mathbb{W}$ -norm is stronger than the  $\mathbb{V}$ -norm, (H4) in general cannot be satisfied.  
**Proof:** Since  $\widehat{A}$  is linear,  $\widehat{B}$  is bilinear,  $\widehat{F}$  is Lipschitz, (H1) is obvious. By (2.20) we see that

$$\begin{aligned} & \langle \widehat{B}(u_1, u_1) - \widehat{B}(u_2, u_2), u_1 - u_2 \rangle \\ &= - \langle \widehat{B}(u_1 - u_2, u_1 - u_2), u_1 \rangle \\ &= - \langle \widehat{B}(u_1 - u_2, u_1 - u_2), u_2 \rangle. \end{aligned}$$

So, by (2.4) (2.14), (2.13), (2.19) and (S1) we have

$$\begin{aligned} & 2 \langle \widehat{\mathcal{A}}(u_1, t) - \widehat{\mathcal{A}}(u_2, t), u_1 - u_2 \rangle \\ &+ |\widehat{\sigma}(u_1, t) - \widehat{\sigma}(u_2, t)|_{\mathbb{V}}^2 + \int_Z |\widehat{f}(u_1, t, z) - \widehat{f}(u_2, t, z)|_{\mathbb{V}}^2 \nu(dz) \\ &\leq -2\mu \langle \widehat{A}(u_1 - u_2), u_1 - u_2 \rangle - 2 \langle \widehat{B}(u_1, u_1) - \widehat{B}(u_2, u_2), u_1 - u_2 \rangle \\ &+ |u_1 - u_2|_{\mathbb{V}}^2 + |\widehat{F}(u_1, t) - \widehat{F}(u_2, t)|_{\mathbb{V}}^2 \\ &+ |\widehat{\sigma}(u_1, t) - \widehat{\sigma}(u_2, t)|_{\mathbb{V}}^2 + \int_Z |\widehat{f}(u_1, t, z) - \widehat{f}(u_2, t, z)|_{\mathbb{V}}^2 \nu(dz) \\ &\leq |u_1 - u_2|_{\mathbb{V}}^2 + 2C_B |u_2|_{\mathbb{W}} |u_1 - u_2|_{\mathbb{V}}^2 + C_A^2 \widetilde{\rho}(t) |u_1 - u_2|_{\mathbb{V}}^2 \\ &\leq \rho(u_2, t) |u_1 - u_2|_{\mathbb{V}}^2. \end{aligned} \tag{4.4}$$

This proves (H2).

By (2.19) and (S2) we get

$$|\widehat{\mathcal{A}}(u, t)|_{\mathbb{W}^*}^2 \leq C \times (|\widehat{A}u|_{\mathbb{W}^*}^2 + |\widehat{B}(u, u)|_{\mathbb{W}^*}^2 + |\widehat{F}(u, t)|_{\mathbb{W}^*}^2) \leq C(K(t) + |u|_{\mathbb{V}}^2 + |u|_{\mathbb{V}}^4),$$

which is (H3).

If  $u \in \mathbb{W}_m$  for some  $m \in \mathbb{N}$ , there exists a constant  $C_m$  such that  $|u|_{\mathbb{W}} \leq C_m |u|_{\mathbb{V}}$ , this will easily yields (H4). ■

By Lemma 4.1, according to Theorem 2 of [14], equation (4.1) has a unique global càdlàg solution  $u^n$  that satisfies the following integral equation:

$$\begin{aligned} u^n(t) = & \Pi_n \xi + \int_0^t \Pi_n \widehat{\mathcal{A}}(u^n(s), s) ds + \int_0^t \Pi_n \widehat{\sigma}(u^n(s), s) dW(s) \\ & + \int_0^t \int_Z \Pi_n \widehat{f}(u^n(s-), s, z) \widetilde{N}(dz ds), \end{aligned} \tag{4.5}$$

for any  $t \in [0, T]$

In order to construct solutions to equation (3.1), we need to establish some a priori estimates for  $u^n$ .



**Lemma 4.2** *Under assumptions of Theorem 3.2, we have*

$$\sup_{n \in \mathbb{N}} E \left( \sup_{t \in [0, T]} |u^n(t)|_{\mathbb{W}}^4 \right) \leq C \times \left( \int_0^T K^2(s) ds + E|\xi|_{\mathbb{W}}^4 \right). \quad (4.6)$$

where  $K(s), s \leq T$  is the function appeared in the condition (S2).

**Proof:** For any given  $n \in \mathbb{N}$ ,  $M > 0$ , we define

$$\tau_M^n := \inf \{t \geq 0 : |u^n(t)|_{\mathbb{W}} \geq M\} \wedge T. \quad (4.7)$$

Since the solution  $u^n$  is càdlàg in  $\mathbb{W}$  and  $\{\mathcal{F}_t\}$ -adapted,  $\tau_M^n$  is a stopping time, and moreover,  $\tau_M^n \uparrow T$   $P$ -a.s. as  $M \rightarrow \infty$ . Note that  $|u^n(t)|_{\mathbb{W}} \leq M$ , for  $t < \tau_M^n$ , so there exists a constant  $C$  such that  $|u^n(t)|_{\mathbb{V}} \leq CM$  whenever  $t < \tau_M^n$ .

By (2.4) and (4.1) for  $i = 1, \dots, n$ , we have

$$\begin{aligned} d(u^n(t), e_i)_{\mathbb{V}} &= d\langle u^n(t), e_i \rangle \\ &= \langle \Pi_n \widehat{\mathcal{A}}(u^n(t), t), e_i \rangle dt + \langle \Pi_n \widehat{\sigma}(u^n(t), t), e_i \rangle dW(t) + \int_Z \langle \Pi_n \widehat{f}(u^n(t-), t, z), e_i \rangle \widetilde{N}(dzdt) \\ &= (\widehat{\mathcal{A}}(u^n(t), t), e_i)_{\mathbb{V}} dt + (\Pi_n \widehat{\sigma}(u^n(t), t), e_i)_{\mathbb{V}} dW(t) + \int_Z (\Pi_n \widehat{f}(u^n(t-), t, z), e_i)_{\mathbb{V}} \widetilde{N}(dzdt). \end{aligned} \quad (4.8)$$

By (2.6) and Lemma 2.2, we know that  $\widehat{\mathcal{A}}(u^n(t), t) \in \mathbb{W}$ , so multiplying both sides of the equation (4.8) by  $\lambda_i$ , we can use (2.5) to obtain

$$d(u^n(t), e_i)_{\mathbb{W}} = (\widehat{\mathcal{A}}(u^n(t), t), e_i)_{\mathbb{W}} dt + (\Pi_n \widehat{\sigma}(u^n(t), t), e_i)_{\mathbb{W}} dW(t) + \int_Z (\Pi_n \widehat{f}(u^n(t-), t, z), e_i)_{\mathbb{W}} \widetilde{N}(dzdt),$$

for  $i = 1, \dots, n$ . Applying Itô formula to  $(u^n(t), e_i)_{\mathbb{W}}^2$  and then summing over  $i$  from 1 to  $n$  yields

$$\begin{aligned} |u^n(t)|_{\mathbb{W}}^2 &= |u^n(0)|_{\mathbb{W}}^2 + 2 \int_0^t (\widehat{\mathcal{A}}(u^n(s), s), u^n(s))_{\mathbb{W}} ds + \int_0^t |\Pi_n \widehat{\sigma}(u^n(s), s)|_{\mathbb{W}}^2 ds \\ &\quad + 2 \int_0^t (\Pi_n \widehat{\sigma}(u^n(s), s), u^n(s))_{\mathbb{W}} dW(s) + 2 \int_0^t \int_Z (\Pi_n \widehat{f}(u^n(s-), s, z), u^n(s-))_{\mathbb{W}} \widetilde{N}(dzds) \\ &\quad + \int_0^t \int_Z |\Pi_n \widehat{f}(u^n(s-), s, z)|_{\mathbb{W}}^2 N(dzds). \end{aligned} \quad (4.9)$$

By (2.11) we have

$$\begin{aligned} &\left( \operatorname{curl} [\Pi (\operatorname{curl} (u^n(s) - \alpha \Delta u^n(s)) \times u^n(s))], \operatorname{curl} (u^n(s) - \alpha \Delta u^n(s)) \right) \\ &= \left( \operatorname{curl} (\operatorname{curl} (u^n(s) - \alpha \Delta u^n(s)) \times u^n(s)), \operatorname{curl} (u^n(s) - \alpha \Delta u^n(s)) \right), \end{aligned} \quad (4.10)$$

Noticing that (see (4.44), (4.45) and (4.46) in [27])

$$\left( \operatorname{curl} (\operatorname{curl} (u^n(s) - \alpha \Delta u^n(s)) \times u^n(s)), \operatorname{curl} (u^n(s) - \alpha \Delta u^n(s)) \right) = 0, \quad (4.11)$$

by (2.12), (4.2), (4.10) and (4.11) we get

$$\begin{aligned}
& (\widehat{\mathcal{A}}(u^n(s), s), u^n(s))_{\mathbb{W}} = \left( \operatorname{curl}[(I + \alpha A)\widehat{\mathcal{A}}(u^n(s), s)], \operatorname{curl}(u^n(s) - \alpha \Delta u^n(s)) \right) \\
& = (\widehat{F}(u^n(s), s), u^n(s))_{\mathbb{W}} + \mu \left( \operatorname{curl}(\Delta u^n(s)), \operatorname{curl}(u^n(s) - \alpha \Delta u^n(s)) \right) \\
& \quad - \left( \operatorname{curl}[\Pi(\operatorname{curl}(u^n(s) - \alpha \Delta u^n(s)) \times u^n(s))], \operatorname{curl}(u^n(s) - \alpha \Delta u^n(s)) \right) \\
& = (\widehat{F}(u^n(s), s), u^n(s))_{\mathbb{W}} - \frac{\mu}{\alpha} |u^n(s)|_{\mathbb{W}}^2 + \frac{\mu}{\alpha} \left( \operatorname{curl}(u^n(s)), \operatorname{curl}(u^n(s) - \alpha \Delta u^n(s)) \right).
\end{aligned} \tag{4.12}$$

Substituting the above equality into (4.9) we have

$$\begin{aligned}
& |u^n(t)|_{\mathbb{W}}^2 + \frac{2\mu}{\alpha} \int_0^t |u^n(s)|_{\mathbb{W}}^2 ds \\
& = |u^n(0)|_{\mathbb{W}}^2 + \frac{2\mu}{\alpha} \int_0^t \left( \operatorname{curl}(u^n(s)), \operatorname{curl}(u^n(s) - \alpha \Delta u^n(s)) \right) ds \\
& \quad + 2 \int_0^t (\widehat{F}(u^n(s), s), u^n(s))_{\mathbb{W}} ds + \int_0^t |\Pi_n \widehat{\sigma}(u^n(s), s)|_{\mathbb{W}}^2 ds \\
& \quad + 2 \int_0^t (\Pi_n \widehat{\sigma}(u^n(s), s), u^n(s))_{\mathbb{W}} dW(s) + 2 \int_0^t \int_Z (\Pi_n \widehat{f}(u^n(s-), s, z), u^n(s-))_{\mathbb{W}} \widetilde{N}(dz ds) \\
& \quad + \int_0^t \int_Z |\Pi_n \widehat{f}(u^n(s-), s, z)|_{\mathbb{W}}^2 N(dz ds).
\end{aligned} \tag{4.13}$$

Applying the Itô formula (cf.[23]) to (4.13) we have

$$\begin{aligned}
& |u^n(t)|_{\mathbb{W}}^4 + \frac{4\mu}{\alpha} \int_0^t |u^n(s)|_{\mathbb{W}}^4 ds \\
& = |u^n(0)|_{\mathbb{W}}^4 + \frac{4\mu}{\alpha} \int_0^t |u^n(s)|_{\mathbb{W}}^2 \left( \operatorname{curl}(u^n(s)), \operatorname{curl}(u^n(s) - \alpha \Delta u^n(s)) \right) ds \\
& \quad + 4 \int_0^t |u^n(s)|_{\mathbb{W}}^2 (\widehat{F}(u^n(s), s), u^n(s))_{\mathbb{W}} ds \\
& \quad + 2 \int_0^t |u^n(s)|_{\mathbb{W}}^2 |\Pi_n \widehat{\sigma}(u^n(s), s)|_{\mathbb{W}}^2 ds + 4 \int_0^t (\Pi_n \widehat{\sigma}(u^n(s), s), u^n(s))_{\mathbb{W}}^2 ds \\
& \quad + 4 \int_0^t |u^n(s)|_{\mathbb{W}}^2 (\Pi_n \widehat{\sigma}(u^n(s), s), u^n(s))_{\mathbb{W}} dW(s) \\
& \quad + 4 \int_0^t \int_Z |u^n(s-)|_{\mathbb{W}}^2 (\Pi_n \widehat{f}(u^n(s-), s, z), u^n(s-))_{\mathbb{W}} \widetilde{N}(dz ds) \\
& \quad + 4 \int_0^t \int_Z (\Pi_n \widehat{f}(u^n(s-), s, z), u^n(s-))_{\mathbb{W}}^2 N(dz ds) \\
& \quad + 2 \int_0^t \int_Z |u^n(s-)|_{\mathbb{W}}^2 |\Pi_n \widehat{f}(u^n(s-), s, z)|_{\mathbb{W}}^2 N(dz ds) \\
& \quad + \int_0^t \int_Z |\Pi_n \widehat{f}(u^n(s-), s, z)|_{\mathbb{W}}^4 N(dz ds) \\
& \quad + 4 \int_0^t \int_Z (\Pi_n \widehat{f}(u^n(s-), s, z), u^n(s-))_{\mathbb{W}} |\Pi_n \widehat{f}(u^n(s-), s, z)|_{\mathbb{W}}^2 N(dz ds).
\end{aligned} \tag{4.14}$$

Set

$$Y^n(r) := \sup_{0 \leq t \leq r \wedge \tau_M^n} |u^n(t)|_{\mathbb{W}}^4.$$

Taking the sup over  $t \leq r \wedge \tau_M^n$ , then taking expectations we get

$$\begin{aligned} EY^n(r) + \frac{4\mu}{\alpha} E \int_0^{r \wedge \tau_M^n} |u^n(s)|_{\mathbb{W}}^4 ds \\ \leq E|u^n(0)|_{\mathbb{W}}^4 + I_1 + I_2 + I_3 + I_4, \end{aligned} \quad (4.15)$$

where

$$\begin{aligned} I_1 &:= \frac{4\mu}{\alpha} E \int_0^{r \wedge \tau_M^n} |u^n(s)|_{\mathbb{W}}^2 \left| \left( \operatorname{curl}(u^n(s)), \operatorname{curl}(u^n(s) - \alpha \Delta u^n(s)) \right) \right| ds \\ &\quad + 4E \int_0^{r \wedge \tau_M^n} |u^n(s)|_{\mathbb{W}}^2 |\widehat{F}(u^n(s), s), u^n(s)|_{\mathbb{W}} ds \\ &\quad + 2E \int_0^{r \wedge \tau_M^n} |u^n(s)|_{\mathbb{W}}^2 |\Pi_n \widehat{\sigma}(u^n(s), s)|_{\mathbb{W}}^2 ds + 4E \int_0^{r \wedge \tau_M^n} (\Pi_n \widehat{\sigma}(u^n(s), s), u^n(s))_{\mathbb{W}}^2 ds; \\ I_2 &:= 4E \sup_{0 \leq t \leq r \wedge \tau_M^n} \left| \int_0^t |u^n(s)|_{\mathbb{W}}^2 (\Pi_n \widehat{\sigma}(u^n(s), s), u^n(s))_{\mathbb{W}} dW(s) \right|; \\ I_3 &:= 4E \sup_{0 \leq t \leq r \wedge \tau_M^n} \left| \int_0^t \int_Z |u^n(s-)|_{\mathbb{W}}^2 (\Pi_n \widehat{f}(u^n(s-), s, z), u^n(s-))_{\mathbb{W}} \widetilde{N}(dz ds) \right|; \\ I_4 &:= E \int_0^{r \wedge \tau_M^n} \int_Z |\mathcal{H}(u^n(s-), s, z)| N(dz ds), \end{aligned}$$

where

$$\begin{aligned} \mathcal{H}(u^n(s-), s, z) &:= 4(\Pi_n \widehat{f}(u^n(s-), s, z), u^n(s-))_{\mathbb{W}}^2 + 2|u^n(s-)|_{\mathbb{W}}^2 |\Pi_n \widehat{f}(u^n(s-), s, z)|_{\mathbb{W}}^2 \\ &\quad + |\Pi_n \widehat{f}(u^n(s-), s, z)|_{\mathbb{W}}^4 + 4(\Pi_n \widehat{f}(u^n(s-), s, z), u^n(s-))_{\mathbb{W}} |\Pi_n \widehat{f}(u^n(s-), s, z)|_{\mathbb{W}}^2. \end{aligned}$$

We now estimate terms  $I_1, I_2, I_3, I_4$ . By the condition (S2) and the inequalities

$$\begin{aligned} |\widehat{F}(u^n(s), s)|_{\mathbb{W}} + |\widehat{\sigma}(u^n(s), s)|_{\mathbb{W}} &\leq C(|F(u^n(s), s)|_{\mathbb{V}} + |\sigma(u^n(s), s)|_{\mathbb{V}}), \\ |u^n(s)|_{\mathbb{W}}^3 |F(u^n(s), s)|_{\mathbb{V}} &\leq \frac{1}{2} |u^n(s)|_{\mathbb{W}}^4 + \frac{1}{2} |u^n(s)|_{\mathbb{W}}^2 |F(u^n(s), s)|_{\mathbb{V}}^2, \\ |\operatorname{curl}(v)|^2 &\leq \frac{2}{\alpha} |v|_{\mathbb{V}}^2, \quad \text{for any } v \in \mathbb{V}, \end{aligned}$$

we have

$$\begin{aligned} I_1 &\leq CE \int_0^{r \wedge \tau_M^n} |u^n(s)|_{\mathbb{W}}^3 |u^n(s)|_{\mathbb{V}} ds + CE \int_0^{r \wedge \tau_M^n} |u^n(s)|_{\mathbb{W}}^3 |F(u^n(s), s)|_{\mathbb{V}} ds \\ &\quad + 6E \int_0^{r \wedge \tau_M^n} |u^n(s)|_{\mathbb{W}}^2 |\widehat{\sigma}(u^n(s), s)|_{\mathbb{W}}^2 ds \\ &\leq CE \int_0^{r \wedge \tau_M^n} |u^n(s)|_{\mathbb{W}}^4 ds + CE \int_0^{r \wedge \tau_M^n} |u^n(s)|_{\mathbb{W}}^2 |F(u^n(s), s)|_{\mathbb{V}}^2 ds \\ &\quad + CE \int_0^{r \wedge \tau_M^n} |u^n(s)|_{\mathbb{W}}^2 |\sigma(u^n(s), s)|_{\mathbb{V}}^2 ds \\ &\leq CE \int_0^{r \wedge \tau_M^n} |u^n(s)|_{\mathbb{W}}^4 ds + C \int_0^T K^2(s) ds. \end{aligned} \quad (4.16)$$

For  $I_2$  and  $I_3$ , we apply BDG inequality (cf. Theorem 1 of [18]) and Young inequality

$$\begin{aligned}
I_2 &\leq CE \left( \int_0^{r \wedge \tau_M^n} |u^n(s)|_{\mathbb{W}}^4 (\Pi_n \widehat{\sigma}(u^n(s), s), u^n(s))_{\mathbb{W}}^2 ds \right)^{\frac{1}{2}} \\
&\leq CE \left[ \sup_{0 \leq s \leq r \wedge \tau_M^n} |u^n(s)|_{\mathbb{W}}^2 \times \left( \int_0^{r \wedge \tau_M^n} (\Pi_n \widehat{\sigma}(u^n(s), s), u^n(s))_{\mathbb{W}}^2 ds \right)^{\frac{1}{2}} \right] \\
&\leq C\varepsilon_2 E \sup_{0 \leq s \leq r \wedge \tau_M^n} |u^n(s)|_{\mathbb{W}}^4 + \frac{C}{\varepsilon_2} E \int_0^{r \wedge \tau_M^n} (\Pi_n \widehat{\sigma}(u^n(s), s), u^n(s))_{\mathbb{W}}^2 ds \\
&\leq C\varepsilon_2 E Y^n(r) + \frac{C}{\varepsilon_2} E \int_0^{r \wedge \tau_M^n} |u^n(s)|_{\mathbb{W}}^2 |\sigma(u^n(s), s)|_{\mathbb{V}}^2 ds \\
&\leq C\varepsilon_2 E Y^n(r) + \frac{C}{\varepsilon_2} \int_0^T K^2(s) ds + \frac{C}{\varepsilon_2} E \int_0^{r \wedge \tau_M^n} |u^n(s)|_{\mathbb{W}}^4 ds;
\end{aligned} \tag{4.17}$$

$$\begin{aligned}
I_3 &\leq CE \left( \int_0^{r \wedge \tau_M^n} \int_Z |u^n(s)|_{\mathbb{W}}^4 (\Pi_n \widehat{f}(u^n(s-), s, z), u^n(s))_{\mathbb{W}}^2 \nu(dz) ds \right)^{\frac{1}{2}} \\
&\leq CE \left[ \sup_{0 \leq s \leq r \wedge \tau_M^n} |u^n(s)|_{\mathbb{W}}^2 \times \left( \int_0^{r \wedge \tau_M^n} \int_Z (\Pi_n \widehat{f}(u^n(s-), s, z), u^n(s))_{\mathbb{W}}^2 \nu(dz) ds \right)^{\frac{1}{2}} \right] \\
&\leq C\varepsilon_3 E \sup_{0 \leq s \leq r \wedge \tau_M^n} |u^n(s)|_{\mathbb{W}}^4 + \frac{C}{\varepsilon_3} E \int_0^{r \wedge \tau_M^n} \int_Z (\Pi_n \widehat{f}(u^n(s-), s, z), u^n(s))_{\mathbb{W}}^2 \nu(dz) ds \\
&\leq C\varepsilon_3 E Y^n(r) + \frac{C}{\varepsilon_3} E \int_0^{r \wedge \tau_M^n} \int_Z |u^n(s)|_{\mathbb{W}}^2 |f(u^n(s), s, z)|_{\mathbb{V}}^2 \nu(dz) ds \\
&\leq C\varepsilon_3 E Y^n(r) + \frac{C}{\varepsilon_3} E \int_0^{r \wedge \tau_M^n} |u^n(s)|_{\mathbb{W}}^2 (K(s) + |u^n(s)|_{\mathbb{V}}^2) ds \\
&\leq C\varepsilon_3 E Y^n(r) + \frac{C}{\varepsilon_3} \int_0^T K^2(s) ds + \frac{C}{\varepsilon_3} E \int_0^{r \wedge \tau_M^n} |u^n(s)|_{\mathbb{W}}^4 ds.
\end{aligned} \tag{4.18}$$

By a straightforward calculation we see that

$$|\mathcal{H}(u^n(s-), s, z)| \leq C |u^n(s-)|_{\mathbb{W}}^2 |f(u^n(s-), s, z)|_{\mathbb{V}}^2 + C |f(u^n(s-), s, z)|_{\mathbb{V}}^4,$$

therefore, by (S2) we have

$$\begin{aligned}
I_4 &= E \int_0^{r \wedge \tau_M^n} \int_Z |\mathcal{H}(u^n(s), s, z)| \nu(dz) ds \\
&\leq CE \int_0^{r \wedge \tau_M^n} \int_Z |u^n(s)|_{\mathbb{W}}^2 |f(u^n(s), s, z)|_{\mathbb{V}}^2 \nu(dz) ds + CE \int_0^{r \wedge \tau_M^n} \int_Z |f(u^n(s), s, z)|_{\mathbb{V}}^4 \nu(dz) ds \\
&\leq CE \int_0^{r \wedge \tau_M^n} |u^n(s)|_{\mathbb{W}}^2 (K(s) + C |u^n(s)|_{\mathbb{V}}^2) ds + CE \int_0^{r \wedge \tau_M^n} (K^2(s) + C |u^n(s)|_{\mathbb{V}}^4) ds \\
&\leq CE \int_0^{r \wedge \tau_M^n} |u^n(s)|_{\mathbb{W}}^4 ds + C \int_0^T K^2(s) ds.
\end{aligned} \tag{4.19}$$

Substituting (4.16)-(4.19) into (4.15), then choosing sufficiently small  $\varepsilon_2, \varepsilon_3$  we obtain

$$E Y^n(r) \leq CE |u^n(0)|_{\mathbb{W}}^4 + C \int_0^T K^2(s) ds + CE \int_0^r Y^n(s) ds. \tag{4.20}$$

Applying Gronwall inequality we have

$$E \sup_{0 \leq t \leq r \wedge \tau_M^n} |u^n(t)|_{\mathbb{W}}^4 \leq C \times \left( \int_0^T K^2(s) ds + E|\xi|_{\mathbb{W}}^4 \right). \quad (4.21)$$

Therefore, letting  $M \rightarrow \infty$ , by the Fatou lemma we get

$$E \sup_{0 \leq t \leq T} |u^n(t)|_{\mathbb{W}}^4 \leq C \times \left( \int_0^T K^2(s) ds + E|\xi|_{\mathbb{W}}^4 \right). \quad (4.22)$$

The proof of Lemma 4.2 is complete. ■

## 4.2 Existence of solutions

**Lemma 4.3** *Under assumptions of Theorem 3.2, there exists a subsequence  $(n_k)$  and an element  $\bar{u} \in L^4(\Omega; L^\infty([0, T]; \mathbb{W}))$  such that*

- (i)  $u^{n_k} \rightharpoonup \bar{u}$  weakly in  $L^2(\Omega \times [0, T]; \mathbb{V})$  and in  $L^2(\Omega; L^\infty([0, T]; \mathbb{V}))$  under weak- $^*$  topology;
- (ii)  $\hat{A}u^{n_k}(s) \rightharpoonup \hat{A}(s)\bar{u}$  in  $L^2(\Omega \times [0, T]; \mathbb{W}^*)$  under weak- $^*$  topology;
- (iii)  $\hat{B}(u^{n_k}(s), u^{n_k}(s)) \rightharpoonup \hat{B}^*(s)$  in  $L^2(\Omega \times [0, T]; \mathbb{W}^*)$  under weak- $^*$  topology;
- (iv)  $\hat{F}(u^{n_k}(s), s) \rightharpoonup \hat{F}^*(s)$  in  $L^2(\Omega \times [0, T]; \mathbb{W}^*)$  under weak- $^*$  topology;
- (v)  $\hat{\sigma}(u^{n_k}(s), s) \rightharpoonup \hat{\sigma}^*(s)$  weakly in  $L^2(\Omega \times [0, T]; \mathbb{V})$ ,

and  $\hat{\sigma}^*(s)$  is  $\{\mathcal{F}_t\}$ -progressively measurable, moreover

$$\int_0^\cdot \hat{\sigma}(u^{n_k}(s), s) dW(s) \rightharpoonup \int_0^\cdot \hat{\sigma}^*(s) dW(s) \text{ in } L^\infty([0, T]; L^2(\Omega, \mathcal{F}_T; \mathbb{V})) \text{ under weak-}^* \text{ topology;}$$

- (vi)  $\hat{f}(u^{n_k}(s-), s, z) \rightharpoonup \hat{f}^*(s, z)$  weakly in  $L^2(\Omega \times [0, T] \times Z; P \times \nu \times dt; \mathbb{V})$ ,

and  $\hat{f}^*(s, z)$  is  $\{\mathcal{F}_t\}$ -predictable, moreover in  $L^\infty([0, T]; L^2(\Omega, \mathcal{F}_T; \mathbb{V}))$  under weak- $^*$  topology,

$$\int_0^\cdot \int_Z \hat{f}(u^{n_k}(s-), s) \tilde{N}(dz ds) \rightharpoonup \int_0^\cdot \int_Z \hat{f}^*(s, z) \tilde{N}(dz ds).$$

**Proof:** (i) is a direct corollary of lemma 4.2. Since a bounded linear operator between two Banach spaces is trivially weakly continuous, so (ii) is derived by (i). By (2.19) and Lemma 4.2 we have

$$E \int_0^T |\hat{B}(u^{n_k}(s), u^{n_k}(s))|_{\mathbb{W}^*}^2 ds \leq CE \int_0^T |u^{n_k}|_{\mathbb{V}}^4 ds \leq C.$$

Therefore the claim (iii) holds. By (S2) and Lemma 4.2 we know that (iv) also holds. The first claim of (v) holds for the the same reason as (iv). The second claim of (v) follows because

$$\left\| \int_0^\cdot \hat{\sigma}(u^{n_k}(s), s) dW(s) \right\|_{L^\infty([0, T]; L^2(\Omega, \mathcal{F}_T; \mathbb{V}))}^2 = \left\| \hat{\sigma}(u^{n_k}(\cdot), \cdot) \right\|_{L^2(\Omega \times [0, T]; \mathbb{V})}^2,$$

and a bounded linear operator between two Banach spaces is weakly continuous. (vi) holds for the same reason as (v). ■

**Theorem 4.1** *Suppose the assumptions of Theorem 3.2 are in place, then there exists a solution to equation (3.1).*

**Proof:** Let  $\bar{u}$  be the process constructed in Lemma 4.3. Set

$$\hat{\mathcal{A}}^*(s) := -\mu \hat{A} \bar{u}(s) - \hat{B}^*(s) + \hat{F}^*(s), \quad (4.23)$$

then we have

$$\hat{\mathcal{A}}(u^{n_k}(s), s) \rightharpoonup \hat{\mathcal{A}}^*(s) \text{ in } L^2(\Omega \times [0, T]; \mathbb{W}^*) \text{ under weak-}^* \text{ topology.}$$

Let us define a  $\mathbb{W}^*$ -valued process  $u$  by

$$u(t) := u_0 + \int_0^t \hat{\mathcal{A}}^*(s) ds + \int_0^t \hat{\sigma}^*(s) dW(s) + \int_0^t \int_Z \hat{f}^*(s, z) \tilde{N}(dz ds), \quad t \in [0, T].$$

We first show that  $u = \bar{u} dt \otimes P$ -a.e.. For all  $w \in \bigcup_{n=1}^\infty \mathbb{W}_n$ ,  $\varphi \in L^\infty([0, T] \times \Omega)$

$$\begin{aligned} & E \int_0^T \langle \bar{u}(t), \varphi(t)w \rangle dt = \lim_{k \rightarrow \infty} E \int_0^T \langle u^{n_k}(t), \varphi(t)w \rangle dt \\ &= \lim_{k \rightarrow \infty} E \int_0^T \langle u^{n_k}(0), \varphi(t)w \rangle dt + \lim_{k \rightarrow \infty} E \int_0^T \int_0^t \langle \Pi_{n_k} \hat{\mathcal{A}}(u^{n_k}(s), s), \varphi(t)w \rangle ds dt \\ & \quad + \lim_{k \rightarrow \infty} E \int_0^T \left\langle \int_0^t \Pi_{n_k} \hat{\sigma}(u^{n_k}(s), s) dW(s), \varphi(t)w \right\rangle dt \\ & \quad + \lim_{k \rightarrow \infty} E \int_0^T \left\langle \int_0^t \int_Z \Pi_{n_k} \hat{f}(u^{n_k}(s-), s, z) \tilde{N}(dz ds), \varphi(t)w \right\rangle dt \\ &= \lim_{k \rightarrow \infty} E \left( (u^{n_k}(0), w)_{\mathbb{V}} \int_0^T \varphi(t) dt \right) + \lim_{k \rightarrow \infty} E \int_0^T \langle \hat{\mathcal{A}}(u^{n_k}(s), s), \int_s^T \varphi(t) dt w \rangle ds \\ & \quad + \lim_{k \rightarrow \infty} \int_0^T E \left( \int_0^t \hat{\sigma}(u^{n_k}(s), s) dW(s), \varphi(t)w \right) dt \\ & \quad + \lim_{k \rightarrow \infty} \int_0^T E \left( \int_0^t \int_Z \hat{f}(u^{n_k}(s-), s, z) \tilde{N}(dz ds), \varphi(t)w \right) dt \\ &= E \int_0^T \langle u(0), \varphi(t)w \rangle dt + E \int_0^T \int_0^t \langle \hat{\mathcal{A}}^*(s), \varphi(t)w \rangle ds dt \\ & \quad + E \int_0^T \left\langle \int_0^t \hat{\sigma}^*(s) dW(s), \varphi(t)w \right\rangle dt + E \int_0^T \left\langle \int_0^t \int_Z \hat{f}^*(s, z) \tilde{N}(dz ds), \varphi(t)w \right\rangle dt \\ &= E \int_0^T \langle u(t), \varphi(t)w \rangle dt. \end{aligned}$$

So, we have  $u = \bar{u} dt \otimes P$ -a.e. and then  $u \in L^4(\Omega; L^\infty([0, T]; \mathbb{W}))$ . By Theorem 2 of [15] we further deduce that  $u$  is a  $\mathbb{V}$ -valued càdlàg  $\mathcal{F}_t$ -adapted process and

$$\begin{aligned} |u(t)|_{\mathbb{V}}^2 &= |u(0)|_{\mathbb{V}}^2 + 2 \int_0^t \langle \hat{\mathcal{A}}^*(s), \bar{u}(s) \rangle ds + \int_0^t |\hat{\sigma}^*(s)|_{\mathbb{V}}^2 ds + 2 \int_0^t (u(s), \hat{\sigma}^*(s))_{\mathbb{V}} dW(s) \\ & \quad + 2 \int_0^t \int_Z (u(s-), \hat{f}^*(s, z))_{\mathbb{V}} \tilde{N}(dz ds) + \int_0^t \int_Z |\hat{f}^*(s, z)|_{\mathbb{V}}^2 N(dz ds). \end{aligned} \quad (4.24)$$

Moreover, By the same arguments as in [33], we can show that the paths of  $u$  are  $\mathbb{W}$ -valued weakly càdlàg. Therefore, in order to prove that  $u$  is a solution of equation (3.1), it suffices to prove that

$$\hat{\mathcal{A}}(u(s), s) = \hat{\mathcal{A}}^*(s), \quad \hat{\sigma}(u(s), s) = \hat{\sigma}^*(s), \quad dt \otimes P - a.e.;$$

$$\widehat{f}(u(s-), s, z) = \widehat{f}^*(s, z) \quad dt \otimes P \otimes \nu - a.e..$$

Recall  $\rho(u, t)$  in (4.3). Let  $\phi$  be a  $\mathbb{V}$ -valued progressively measurable process and  $\phi \in L^4(\Omega; L^\infty([0, T]; \mathbb{W}))$ , applying Itô formula (cf.[23]) we have

$$\begin{aligned} & e^{-\int_0^t \rho(\phi(s), s) ds} |u^{n_k}(t)|_{\mathbb{V}}^2 \\ &= |u^{n_k}(0)|_{\mathbb{V}}^2 + \int_0^t e^{-\int_0^s \rho(\phi(r), r) dr} \left[ 2\langle \widehat{\mathcal{A}}(u^{n_k}(s), s), u^{n_k}(s) \rangle \right. \\ & \quad \left. + |\Pi_{n_k} \widehat{\sigma}(u^{n_k}(s), s)|_{\mathbb{V}}^2 - \rho(\phi(s), s) |u^{n_k}(s)|_{\mathbb{V}}^2 \right] ds \\ & \quad + 2 \int_0^t e^{-\int_0^s \rho(\phi(r), r) dr} (\widehat{\sigma}(u^{n_k}(s), s), u^{n_k}(s))_{\mathbb{V}} dW(s) \\ & \quad + 2 \int_0^t \int_Z e^{-\int_0^s \rho(\phi(r), r) dr} (\widehat{f}(u^{n_k}(s-), s, z), u^{n_k}(s-))_{\mathbb{V}} \widetilde{N}(dz ds) \\ & \quad + 2 \int_0^t \int_Z e^{-\int_0^s \rho(\phi(r), r) dr} |\Pi_{n_k} \widehat{f}(u^{n_k}(s-), s, z)|_{\mathbb{V}}^2 N(dz ds). \end{aligned}$$

Taking expectations on both sides of the above equality and by the local monotonicity in lemma 4.1 we get

$$E \left( e^{-\int_0^t \rho(\phi(s), s) ds} |u^{n_k}(t)|_{\mathbb{V}}^2 \right) - E |u^{n_k}(0)|_{\mathbb{V}}^2 \leq EI_1(t) + EI_2(t) \leq EI_2(t),$$

where

$$\begin{aligned} I_1(t) &:= \int_0^t e^{-\int_0^s \rho(\phi(r), r) dr} \left( 2\langle \widehat{\mathcal{A}}(u^{n_k}(s), s) - \widehat{\mathcal{A}}(\phi(s), s), u^{n_k}(s) - \phi(s) \rangle \right. \\ & \quad \left. + |\widehat{\sigma}(u^{n_k}(s), s) - \widehat{\sigma}(\phi(s), s)|_{\mathbb{V}}^2 + \int_Z |\widehat{f}(u^{n_k}(s), s, z) - \widehat{f}(\phi(s), s, z)|_{\mathbb{V}}^2 \nu(dz) \right. \\ & \quad \left. - \rho(\phi(s), s) |u^{n_k}(s) - \phi(s)|_{\mathbb{V}}^2 \right) ds \leq 0; \end{aligned}$$

$$\begin{aligned} I_2(t) &:= \int_0^t e^{-\int_0^s \rho(\phi(r), r) dr} \left( 2\langle \widehat{\mathcal{A}}(u^{n_k}(s), s) - \widehat{\mathcal{A}}(\phi(s), s), \phi(s) \rangle \right. \\ & \quad \left. + 2\langle \widehat{\mathcal{A}}(\phi(s), s), u^{n_k}(s) \rangle - |\widehat{\sigma}(\phi(s), s)|_{\mathbb{V}}^2 + 2(\widehat{\sigma}(u^{n_k}(s), s), \widehat{\sigma}(\phi(s), s))_{\mathbb{V}} \right. \\ & \quad \left. + 2 \int_Z (\widehat{f}(u^{n_k}(s), s, z), \widehat{f}(\phi(s), s, z))_{\mathbb{V}} \nu(dz) - \int_Z |\widehat{f}(\phi(s), s, z)|_{\mathbb{V}}^2 \nu(dz) \right. \\ & \quad \left. - 2\rho(\phi(s), s) (u^{n_k}(s), \phi(s))_{\mathbb{V}} + \rho(\phi(s), s) |\phi(s)|_{\mathbb{V}}^2 \right) ds. \end{aligned}$$

Since for any nonnegative function  $\psi \in L^\infty([0, T])$

$$\begin{aligned} & E \int_0^T \psi(t) |\overline{u}(t)|_{\mathbb{V}}^2 dt = \lim_{k \rightarrow \infty} E \int_0^T (\psi(t) \overline{u}(t), u^{n_k}(t))_{\mathbb{V}} dt \\ & \leq \left( E \int_0^T \psi(t) |\overline{u}(t)|_{\mathbb{V}}^2 dt \right)^{\frac{1}{2}} \liminf_{k \rightarrow \infty} \left( E \int_0^T \psi(t) |u^{n_k}(t)|_{\mathbb{V}}^2 dt \right)^{\frac{1}{2}}, \end{aligned} \quad (4.25)$$

noticing that  $u = \overline{u} \quad dt \otimes P$ -a.e., we have

$$E \int_0^T \psi(t) |u(t)|_{\mathbb{V}}^2 dt \leq \liminf_{k \rightarrow \infty} E \int_0^T \psi(t) |u^{n_k}(t)|_{\mathbb{V}}^2 dt.$$

Using the Fubini theorem, then letting  $k \rightarrow \infty$  and by Lemma 4.3 we obtain that

$$\begin{aligned}
& E \int_0^T \psi(t) \left[ e^{-\int_0^t \rho(\phi(s), s) ds} |u(t)|_{\mathbb{V}}^2 - |u(0)|_{\mathbb{V}}^2 \right] dt \\
& \leq \liminf_{k \rightarrow \infty} E \int_0^T \psi(t) \left[ e^{-\int_0^t \rho(\phi(s), s) ds} |u^{n_k}(t)|_{\mathbb{V}}^2 - |u^{n_k}(0)|_{\mathbb{V}}^2 \right] dt \\
& \leq \liminf_{k \rightarrow \infty} E \int_0^T \psi(t) I_2(t) dt \\
& \leq E \int_0^T \psi(t) \int_0^t e^{-\int_0^s \rho(\phi(r), r) dr} \left[ 2\langle \hat{\mathcal{A}}^*(s) - \hat{\mathcal{A}}(\phi(s), s), \phi(s) \rangle \right. \\
& \quad + 2\langle \hat{\mathcal{A}}(\phi(s), s), \bar{u}(s) \rangle - |\hat{\sigma}(\phi(s), s)|_{\mathbb{V}}^2 + 2(\hat{\sigma}^*(s), \hat{\sigma}(\phi(s), s))_{\mathbb{V}} \\
& \quad + 2 \int_Z (\hat{f}^*(s, z), \hat{f}(\phi(s), s, z))_{\mathbb{V}} \nu(dz) - \int_Z |\hat{f}(\phi(s), s, z)|_{\mathbb{V}}^2 \nu(dz) \\
& \quad \left. - 2\rho(\phi(s), s)(\bar{u}(s), \phi(s))_{\mathbb{V}} + \rho(\phi(s), s)|\phi(s)|_{\mathbb{V}}^2 \right] ds dt. \tag{4.26}
\end{aligned}$$

On the other hand, by (4.24) and Itô formula,

$$\begin{aligned}
& E \left( e^{-\int_0^t \rho(\phi(s), s) ds} |u(t)|_{\mathbb{V}}^2 \right) - E |u(0)|_{\mathbb{V}}^2 \\
& = E \int_0^t e^{-\int_0^s \rho(\phi(r), r) dr} \left[ 2\langle \hat{\mathcal{A}}^*(s), \bar{u}(s) \rangle + |\hat{\sigma}^*(s)|_{\mathbb{V}}^2 + \int_Z |\hat{f}^*(s, z)|_{\mathbb{V}}^2 \nu(dz) - \rho(\phi(s), s)|u(s)|_{\mathbb{V}}^2 \right] ds.
\end{aligned}$$

Hence, by the Fubini theorem we get

$$\begin{aligned}
& E \int_0^T \psi(t) \left[ e^{-\int_0^t \rho(\phi(s), s) ds} |u(t)|_{\mathbb{V}}^2 - |u(0)|_{\mathbb{V}}^2 \right] dt \\
& = E \int_0^T \psi(t) \int_0^t e^{-\int_0^s \rho(\phi(r), r) dr} \left[ 2\langle \hat{\mathcal{A}}^*(s), \bar{u}(s) \rangle + |\hat{\sigma}^*(s)|_{\mathbb{V}}^2 \right. \\
& \quad \left. + \int_Z |\hat{f}^*(s, z)|_{\mathbb{V}}^2 \nu(dz) - \rho(\phi(s), s)|u(s)|_{\mathbb{V}}^2 \right] ds dt. \tag{4.27}
\end{aligned}$$

Combining (4.26) and (4.27) we arrive at

$$\begin{aligned}
& E \int_0^T \psi(t) \int_0^t e^{-\int_0^s \rho(\phi(r), r) dr} \left[ 2\langle \hat{\mathcal{A}}^*(s) - \hat{\mathcal{A}}(\phi(s), s), \bar{u}(s) - \phi(s) \rangle + |\hat{\sigma}^*(s) - \hat{\sigma}(\phi(s), s)|_{\mathbb{V}}^2 \right. \\
& \quad \left. + \int_Z |\hat{f}^*(s, z) - \hat{f}(\phi(s), s, z)|_{\mathbb{V}}^2 \nu(dz) - \rho(\phi(s), s)|\bar{u}(s) - \phi(s)|_{\mathbb{V}}^2 \right] ds dt \leq 0. \tag{4.28}
\end{aligned}$$

If we put  $\phi(s) = \bar{u}(s)$  in (4.28), we obtain

$$\begin{aligned}
& \hat{\sigma}^*(s) = \hat{\sigma}(\bar{u}(s), s) = \hat{\sigma}(u(s), s) \quad \text{in } L^2(\Omega \times [0, T]; \mathbb{V}); \\
& \hat{f}^*(s, z) = \hat{f}(\bar{u}(s), s, z) = \hat{f}(u(s), s, z) = \hat{f}(u(s-), s, z) \quad \text{in } L^2(\Omega \times [0, T] \times Z; P \times \nu \times dt; \mathbb{V}).
\end{aligned}$$

Thus, (4.28) implies that

$$\begin{aligned}
& E \int_0^T \psi(t) \int_0^t e^{-\int_0^s \rho(\phi(r), r) dr} \left[ 2\langle \hat{\mathcal{A}}^*(s) - \hat{\mathcal{A}}(\phi(s), s), \bar{u}(s) - \phi(s) \rangle \right. \\
& \quad \left. - \rho(\phi(s), s)|\bar{u}(s) - \phi(s)|_{\mathbb{V}}^2 \right] ds dt \leq 0. \tag{4.29}
\end{aligned}$$



Putting  $\phi = \bar{u} - \varepsilon \tilde{\phi} w$  in (4.29) for  $\tilde{\phi} \in L^\infty([0, T] \times \Omega; dt \otimes P)$  and  $w \in \mathbb{W}$ , then dividing both sides by  $\varepsilon$  and letting  $\varepsilon \rightarrow 0$ , we finally have

$$E \int_0^T \psi(t) \int_0^t e^{-\int_0^s \rho(\bar{u}(r), r) dr} \left[ 2\tilde{\phi}(s) \langle \hat{\mathcal{A}}^*(s) - \hat{\mathcal{A}}(\bar{u}(s), s), w \rangle \right] ds dt \leq 0.$$

Since  $\tilde{\phi}$  is arbitrary,

$$\hat{\mathcal{A}}^*(s) = \hat{\mathcal{A}}(\bar{u}(s), s) = \hat{\mathcal{A}}(u(s), s) \quad \text{in } L^2(\Omega \times [0, T]; \mathbb{W}^*).$$

Therefore, we conclude that the process  $u$  is a solution of equation (3.1). ■

### 4.3 Uniqueness of solutions

We finally proceed to show the uniqueness of solutions to equation (3.1), thus completes the proof of Theorem 3.2.

Suppose that  $u = \{u(t)\}$  and  $v = \{v(t)\}$  are two solutions of (3.1) with initial conditions  $u_0, v_0$  respectively, i.e.

$$\begin{aligned} u(t) &= u_0 + \int_0^t \hat{\mathcal{A}}(u(s), s) ds + \int_0^t \hat{\sigma}(u(s), s) dW(s) + \int_0^t \int_Z \hat{f}(u(s-), s, z) \tilde{N}(ds dz), \quad t \in [0, T]; \\ v(t) &= v_0 + \int_0^t \hat{\mathcal{A}}(v(s), s) ds + \int_0^t \hat{\sigma}(v(s), s) dW(s) + \int_0^t \int_Z \hat{f}(v(s-), s, z) \tilde{N}(ds dz), \quad t \in [0, T]. \end{aligned}$$

We define the stopping time

$$\tau_N := \inf\{t \in [0, T] : |u(t)|_{\mathbb{V}} \geq N\} \wedge \inf\{t \in [0, T] : |v(t)|_{\mathbb{V}} \geq N\} \wedge T.$$

Using the Itô formula (cf.[23]) we have

$$\begin{aligned} & e^{-\int_0^{t \wedge \tau_N} \rho(v(s), s) ds} |u(t \wedge \tau_N) - v(t \wedge \tau_N)|_{\mathbb{V}}^2 \\ &= |u_0 - v_0|_{\mathbb{V}}^2 + \int_0^{t \wedge \tau_N} e^{-\int_0^s \rho(v(r), r) dr} \left[ 2\langle \hat{\mathcal{A}}(u(s), s) - \hat{\mathcal{A}}(v(s), s), u(s) - v(s) \rangle \right. \\ &\quad \left. + |\hat{\sigma}(u(s), s) - \hat{\sigma}(v(s), s)|_{\mathbb{V}}^2 - \rho(v(s), s) |u(s) - v(s)|_{\mathbb{V}}^2 \right] ds \\ &\quad + 2 \int_0^{t \wedge \tau_N} e^{-\int_0^s \rho(v(r), r) dr} (\hat{\sigma}(u(s), s) - \hat{\sigma}(v(s), s), u(s) - v(s))_{\mathbb{V}} dW(s) \\ &\quad + 2 \int_0^{t \wedge \tau_N} \int_Z e^{-\int_0^s \rho(v(r), r) dr} (\hat{f}(u(s-), s, z) - \hat{f}(v(s-), s, z), u(s-) - v(s-))_{\mathbb{V}} \tilde{N}(dz ds) \\ &\quad + 2 \int_0^{t \wedge \tau_N} \int_Z e^{-\int_0^s \rho(v(r), r) dr} |\hat{f}(u(s-), s, z) - \hat{f}(v(s-), s, z)|_{\mathbb{V}}^2 N(dz ds). \end{aligned}$$

It then follows from the local monotonicity (H2) of Lemma 4.1 that

$$\begin{aligned} & E \left[ e^{-\int_0^{t \wedge \tau_N} \rho(v(s), s) ds} |u(t \wedge \tau_N) - v(t \wedge \tau_N)|_{\mathbb{V}}^2 \right] - E |u_0 - v_0|_{\mathbb{V}}^2 \\ &= E \left[ \int_0^{t \wedge \tau_N} e^{-\int_0^s \rho(v(r), r) dr} \left( 2\langle \hat{\mathcal{A}}(u(s), s) - \hat{\mathcal{A}}(v(s), s), u(s) - v(s) \rangle \right. \right. \end{aligned}$$

$$\begin{aligned}
& + |\widehat{\sigma}(u(s), s) - \widehat{\sigma}(v(s), s)|_{\mathbb{V}}^2 - \rho(v(s), s)|u(s) - v(s)|_{\mathbb{V}}^2 \\
& + \int_Z e^{-\int_0^s \rho(v(r), r) dr} |\widehat{f}(u(s), s, z) - \widehat{f}(v(s), s, z)|_{\mathbb{V}}^2 \nu(dz) ds \Big] \\
& \leq 0.
\end{aligned}$$

Hence if  $u_0 = v_0$   $P$ -a.e., then

$$E \left[ e^{-\int_0^{t \wedge \tau_N} \rho(v(s), s) ds} |u(t \wedge \tau_N) - v(t \wedge \tau_N)|_{\mathbb{V}}^2 \right] = 0, \quad t \in [0, T].$$

By (4.3) we get

$$\int_0^T \rho(v(s), s) ds < \infty, \quad P\text{-a.s..}$$

Therefore, by letting  $N \rightarrow \infty$  we have  $u(t) = v(t)$   $P$ -a.s.. The pathwise uniqueness follows from the càdlàg property of  $u, v$  in the solutions.

This completes the proof of Theorem 3.2.

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## References

- [1] Z. Brzeźniak, E. Hausenblas, and J. Zhu. 2D stochastic Navier-Stokes equations driven by jump noise. *Nonlinear Analysis: Theory, Methods & Applications*, 79:122–139, 2013.
- [2] Z. Brzeźniak, W. Liu, and J. Zhu. Strong solutions for SPDE with locally monotone coefficients driven by Lévy noise. *Nonlinear Analysis: Real World Applications*, 17:283–310, 2014.
- [3] A. V. Busuioc and T. S. Ratiu. The second grade fluid and averaged Euler equations with Navier-slip boundary conditions. *Nonlinearity*, 16(3):1119, 2003.
- [4] D. Cioranescu and V. Girault. Weak and classical solutions of a family of second grade fluids. *International Journal of Non-Linear Mechanics*, 32(2):317–335, 1997.
- [5] D. Cioranescu and E. H. Ouazar. Existence and uniqueness for fluids of second grade. *Nonlinear Partial Differential Equations*, 109:178–197, 1984.
- [6] Z. Dong, W. V. Li, and J. Zhai. Stationary weak solutions for stochastic 3D Navier-Stokes equations with Lévy noise. *Stochastics and Dynamics*, 12(01):1150006, 2012.
- [7] Z. Dong and Y. Xie. Global solutions of stochastic 2D Navier-Stokes equations with Lévy noise. *Science in China Series A: Mathematics*, 52(7):1497–1524, 2009.
- [8] Z. Dong and J. Zhai. Martingale solutions and Markov selection of stochastic 3D Navier-Stokes equations with jump. *Journal of Differential Equations*, 250(6):2737–2778, 2011.
- [9] J. Dunn and K. Rajagopal. Fluids of differential type: critical review and thermodynamic analysis. *International Journal of Engineering Science*, 33(5):689–729, 1995.

- [10] J. E. Dunn and R. L. Fosdick. Thermodynamics, stability, and boundedness of fluids of complexity 2 and fluids of second grade. *Archive for Rational Mechanics and Analysis*, 56(3):191–252, 1974.
- [11] R. Fosdick and K. Rajagopal. Anomalous features in the model of second order fluids. *Archive for Rational Mechanics and Analysis*, 70(2):145–152, 1979.
- [12] B. Guo, C. Guo, and J. Zhang. Martingale and stationary solutions for stochastic non-Newtonian fluids. *Differential Integral Equations*, 23(3/4):303–326, 03 2010.
- [13] I. Gyöngy. On stochastic equations with respect to semimartingales iii. *Stochastics*, 7(4):231–254, 1982.
- [14] I. Gyöngy and N. V. Krylov. On stochastic equations with respect to semimartingales i. *Stochastics*, 4(1):1–21, 1980.
- [15] I. Gyöngy and N. V. Krylov. On stochastic equations with respect to semimartingales ii. itô formula in banach spaces. *Stochastics*, 6(3-4):153–173, 1982.
- [16] E. Hausenblas and P. A. Razafimandimby. On stochastic evolution equations for non-linear bipolar fluids: Well-posedness and some properties of the solution. *Journal of Mathematical Analysis and Applications*, 441(2):763 – 800, 2016.
- [17] E. Hausenblas, P. A. Razafimandimby, and M. Sango. Martingale Solution to Equations for Differential Type Fluids of Grade Two Driven by Random Force of Lévy Type. *Potential Analysis*, 38(4):1291–1331, 2013.
- [18] A. Ichikawa. Some inequalities for martingales and stochastic convolutions. *Stochastic Analysis and Applications*, 4(3):329–339, 1986.
- [19] D. Iftimie. Remarques sur la limite  $\alpha \rightarrow 0$  pour les fluides de grade 2. *Studies in Mathematics and its Applications*, 31:457–468, 2002.
- [20] N. V. Krylov and B. L. Rozovskii. Stochastic evolution equations. *Journal of Soviet Mathematics*, 16(4):1233–1277, 1981.
- [21] W. Liu and M. Röckner. SPDE in Hilbert space with locally monotone coefficients. *Journal of Functional Analysis*, 259(11):2902–2922, 2010.
- [22] W. Liu and M. Röckner. Local and global well-posedness of SPDE with generalized coercivity conditions. *Journal of differential equations*, 254(2):725–755, 2013.
- [23] M. Métivier. *Semimartingales: a course on stochastic processes*, volume 2. Walter de Gruyter, 1982.
- [24] I. Moise, R. Rosa, and X. Wang. Attractors for non-compact semigroups via energy equations. *Nonlinearity*, 11(5):1369, 1998.
- [25] E. Pardou. *Equations aux dérivées partielles stochastiques non linéaires monotones*. PhD thesis, Université Paris XI, 1975.
- [26] C. Prévôt and M. Röckner. *A concise course on stochastic partial differential equations*, volume 1905. Springer, 2007.

- [27] P. A. Razafimandimby and M. Sango. Weak solutions of a stochastic model for two-dimensional second grade fluids. *Boundary Value Problems*, 2010(1):1–47, 2010.
- [28] P. A. Razafimandimby and M. Sango. Strong solution for a stochastic model of two-dimensional second grade fluids: Existence, uniqueness and asymptotic behavior. *Nonlinear Analysis: Theory, Methods & Applications*, 75(11):4251 – 4270, 2012.
- [29] J. Ren, M. Röckner, and F.-Y. Wang. Stochastic generalized porous media and fast diffusion equations. *Journal of Differential Equations*, 238(1):118–152, 2007.
- [30] R. S. Rivlin and J. Ericksen. Stress-deformation relations for isotropic materials. In *Collected Papers of RS Rivlin*, pages 911–1013. Springer, 1997.
- [31] S. Shkoller. Smooth global Lagrangian flow for the 2D Euler and second-grade fluid equations. *Applied Mathematics Letters*, 14(5):539 – 543, 2001.
- [32] V. Solonnikov. On general boundary problems for systems which are elliptic in the sense of A. Douglis and L. Nirenberg. II. *Proceedings of the Steklov Institute of Mathematics*, 92:269–339, 1968.
- [33] R. Temam. *Navier-Stokes equations: theory and numerical analysis*, volume 343. American Mathematical Soc., 2001.
- [34] C. Truesdell and W. Noll. The non-linear field theories of mechanics. In *The non-linear field theories of mechanics*, pages 1–579. Springer, 2004.
- [35] R. Wang, J. Zhai, and T. Zhang. A moderate deviation principle for 2-D stochastic Navier-Stokes equations. *Journal of Differential Equations*, 258(10):3363 – 3390, 2015.
- [36] R. Wang, J. Zhai, and T. Zhang. Exponential mixing for stochastic model of two-dimensional second grade fluids. *Nonlinear Analysis: Theory, Methods & Applications*, 132:196–213, 2016.
- [37] T. Xu and T. Zhang. Large deviation principles for 2-D stochastic Navier–Stokes equations driven by Lévy processes. *Journal of Functional Analysis*, 257(5):1519–1545, 2009.
- [38] J. Zhai and T. Zhang. Large deviations for 2-D stochastic Navier-Stokes equations driven by multiplicative Lévy noises. *Bernoulli*, 21(4):2351–2392, 11 2015.
- [39] J. Zhai and T. Zhang. Large deviations for stochastic models of two-dimensional second grade fluids. *Appl. Math. Optimiz.*, 1–28, 2015.
- [40] J. Zhai, T. Zhang and W. Zheng, Moderate deviations for stochastic models of two-dimensional second grade fluids. arXiv: 1607.08669